

Non-geometric background arising in the solution of Neumann boundary conditions *

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March 3, 2013

Abstract

We investigate the open string propagation in the weakly curved background with the Kalb-Ramond field containing an infinitesimal part, linear in coordinate. Solving the Neumann boundary conditions, we find the expression for the space-time coordinates in terms of the effective ones. So, the initial theory reduces to the effective one. This effective theory is defined on the non-geometric doubled space (q^μ, \tilde{q}_μ) , where q^μ is the effective coordinate and \tilde{q}_μ is its T-dual. The effective metric depends on the coordinate q^μ and there exists non-trivial effective Kalb-Ramond field which depends on the T-dual coordinate \tilde{q}_μ . The fact that \tilde{q}_μ is Ω -odd leads to the nonvanishing effective Kalb-Ramond field.

1 Introduction

Recently, a formulation of the string theory in the non-geometric backgrounds has been considered [1]-[5]. In the geometric background the transition functions between the coordinate patches are the fundamental symmetries of the theory as the diffeomorphisms and the gauge transformations. But, what if the new fundamental symmetry, the T-duality is promoted to the transition function? Then one talks about the non-geometric T-folds and the string theory on such background can be consistently defined [1].

Because T-duality mixes the momentum and the winding modes, the new transition function will relate the patches parametrized by both the conventional coordinates x^μ

*Work supported in part by the Serbian Ministry of Education and Science, under contract No. 171031.

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(conjugate to the momenta modes) and their T-duals \tilde{x}^μ (conjugate to the winding modes). The theory is formulated in the doubled formalism with coordinates (x^μ, \tilde{x}^μ) . In order to preserve the number of degrees of freedom some relation between x^μ and \tilde{x}^μ must be imposed. So, while the geometric backgrounds can depend on the space-time coordinates, the non-geometric backgrounds have the nontrivial dependence on the dual coordinates as well [1].

Non-geometric backgrounds arise in many different approaches. They are often connected with the T-duality on NS-NS fluxes [2], and compactification with duality twists [3]. There also exists the formulation of the double field theory [4] as the field theory on doubled torus. More details and extended list of references can be found in the review paper [5].

In the present paper, we present yet another case where the non-geometric space naturally appears. In comparison with other investigations, ours do not include compactified coordinates nor do we perform the T-duality. We find the solution of Neumann boundary conditions on the open string endpoints, which turns the weakly curved background of the original theory into the doubled background of the effective theory.

We start with the theory describing the motion of the open bosonic string in the background created by the closed string modes $G_{\mu\nu}$ and $B_{\mu\nu}$. In our previous papers [6] and [7], we treated the Neumann boundary conditions as constraints and used the Dirac prescription. In the present paper we show that presupposing the appropriate form of the solution, its explicit final expression may be obtained in a much simpler way. On this solution of boundary condition, we obtain the effective theory. The effective background fields are the background fields seen by the effective string. In the previously investigated cases, of the flat background ($b_{\mu\nu} \neq 0$ and $B_{\mu\nu\rho} = 0$) and the weakly curved one without constant term of Kalb-Ramond field ($b_{\mu\nu} = 0$ and $B_{\mu\nu\rho} \neq 0$ [6]) the effective metric is constant and the effective Kalb-Ramond field is zero.

In the present case of a weakly curved background ($b_{\mu\nu} \neq 0$ and $B_{\mu\nu\rho} \neq 0$) the situation is quite different. The effective theory is defined on the doubled target space (q^μ, \tilde{q}^μ) . These effective coordinates appear naturally in the solution of the Neumann boundary conditions. Moreover, the second coordinate of the target space turns out to be the T-dual of the first one. The effective metric depends on the effective coordinate q^μ , which is symmetric under σ -parity transformation $\Omega : \sigma \rightarrow -\sigma$ and the effective Kalb-Ramond field depends on the \tilde{q}^μ , which is antisymmetric. Because of this fact the term in the action containing the effective Kalb-Ramond field becomes Ω -even, which allows its survival. In the conventional space, with the effective coordinate q^μ , the effective Kalb-Ramond field comes within the Ω -odd term in the action and as such is eliminated. In the present paper, because of the fact that the effective theory is defined in the doubled space, the effective string can actually see the effective Kalb-Ramond field.

2 Open string theory in weakly curved background

We are considering propagation of the open bosonic string, in the background defined by the metric tensor $G_{\mu\nu}$ and Kalb-Ramond antisymmetric field $B_{\mu\nu}$. It is described by the action [8, 9, 10, 11, 12]

$$S = \kappa \int_{\Sigma} d^2\xi \left[\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}(x) \right] \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu}, \quad (2.1)$$

($\epsilon^{01} = -1$), where integration goes over two-dimensional world-sheet Σ parametrized by the coordinates $\xi^0 = \tau$, $\xi^1 = \sigma$ with $\sigma \in [0, \pi]$. Here $x^{\mu}(\xi)$, $\mu = 0, 1, \dots, D-1$ are the coordinates of the D-dimensional space-time, and we use the notation $\dot{x} = \frac{\partial x}{\partial \tau}$, $x' = \frac{\partial x}{\partial \sigma}$. The requirement for the world-sheet conformal invariance on the quantum level, produces the restriction on the background fields. They must obey the space-time equations of motion. To the lowest order in slope parameter α' , for the constant dilaton field $\Phi = \text{const}$ these equations have the form

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0, \quad (2.2)$$

$$D_{\rho} B^{\rho}{}_{\mu\nu} = 0, \quad (2.3)$$

where $B_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} + \partial_{\nu} B_{\rho\mu} + \partial_{\rho} B_{\mu\nu}$ is a field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_{μ} are Ricci tensor and covariant derivative with respect to space-time metric. We will consider the following particular solution of these equations, the *weakly curved* background [6, 7, 13, 14]

$$G_{\mu\nu} = \text{const}, \quad B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^{\rho}, \quad (2.4)$$

where the parameter $b_{\mu\nu}$ is constant and $B_{\mu\nu\rho}$ is constant and infinitesimally small. Through the paper we will work up to the first order in $B_{\mu\nu\rho}$.

The minimal action principle for the open string leads to the equation of motion

$$\ddot{x}^{\mu} = x''^{\mu} - 2B^{\mu}{}_{\nu\rho} \dot{x}^{\nu} x'^{\rho}, \quad (2.5)$$

and the boundary conditions on the string endpoints. Choosing the Neumann boundary conditions we have

$$\gamma_{\mu}^0 \Big|_{\sigma=0,\pi} = 0, \quad (2.6)$$

where

$$\gamma_{\mu}^0 \equiv \frac{\delta \mathcal{L}}{\delta x'^{\mu}} = G_{\mu\nu} x'^{\nu} - 2B_{\mu\nu} \dot{x}^{\nu}. \quad (2.7)$$

Our goal is to solve the boundary conditions and obtain the effective theory on this solution. We achieve this, by imposing the ansatz for the solution and by demanding that it obeys the equations of motion and the consistency condition.

2.1 Solution of the boundary conditions

It is well known that in the case of the constant background fields, both equations of motion and boundary conditions can be solved by expressing odd coordinate part \bar{q}^μ in terms of the even coordinate part q^μ . We are going to generalize such a solution to the case of the weakly curved background.

Let us define the even and odd coordinate parts with respect to $\sigma = 0$

$$\begin{aligned} q^\mu(\sigma) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu} \Big|_{\sigma=0}, \\ \bar{q}^\mu(\sigma) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu} \Big|_{\sigma=0}, \end{aligned} \quad (2.8)$$

and separate the even and odd parts of the equation of motion (2.5) and the boundary condition (2.6) at $\sigma = 0$. In this way, the equations of motion become

$$\begin{aligned} \ddot{q}^\mu - q''^\mu &= -2B_{\nu\rho}^\mu \left[\dot{q}^\nu \bar{q}'^\rho + \dot{\bar{q}}^\nu q'^\rho \right], \\ \ddot{\bar{q}}^\mu - \bar{q}''^\mu &= -2B_{\nu\rho}^\mu \left[\dot{q}^\nu q'^\rho + \dot{\bar{q}}^\nu \bar{q}'^\rho \right], \end{aligned} \quad (2.9)$$

and only the even part of γ_μ^0 contributes to the boundary conditions at $\sigma = 0$

$$\begin{aligned} \gamma_\mu^0 \Big|_{\sigma=0} &= 0, \\ \gamma_\mu^0 &\equiv G_{\mu\nu} \bar{q}'^\nu - 2b_{\mu\nu} \dot{q}^\nu - 2h_{\mu\nu}(q) \dot{q}^\nu, \end{aligned} \quad (2.10)$$

with $h_{\mu\nu}(x) = B_{\mu\nu}(x) - b_{\mu\nu}$ being infinitesimally small.

Let us look for the solution of this boundary condition, in the form

$$\begin{aligned} \dot{q}^\mu &= -A_{1\nu}^\mu(\tilde{q}) \dot{q}^\nu + 2\beta_{1\nu}^\mu(q) q'^\nu, \\ \dot{\bar{q}}^\mu &= -A_{2\nu}^\mu(\tilde{q}) q'^\nu + 2\beta_{2\nu}^\mu(q) \dot{q}^\nu. \end{aligned} \quad (2.11)$$

We suppose that the first derivatives in τ and σ of \bar{q} are the linear functions of the first derivatives of q , and that they do not include the higher derivative terms. The characteristics of the coefficients A and β arguments, are dictated by the parity of both equations. The coefficients $A_{1\nu}^\mu$ and $A_{2\nu}^\mu$ are odd and as such they do not contain constant terms and they depend on some odd variable \tilde{q} , while $\beta_{1\nu}^\mu$ and $\beta_{2\nu}^\mu$ are even functions, depending on the (new independent) variable q^μ . Beside satisfying (2.10), the solution (2.11) must obey the equations of motion (2.9), the consistency condition $(\dot{\bar{q}}^\mu)' = (\bar{q}'^\mu)'$ and it must be in agreement with the zeroth order solution

$$\dot{q}^\mu = 2b_{\nu}^\mu q'^\nu, \quad \dot{\bar{q}}^\mu = 2b_{\nu}^\mu \dot{q}^\nu, \quad (2.12)$$

for $B_{\mu\nu\rho} = 0$.

The equations of motion (2.9) and the zeroth order solution (2.12) are invariant to the interchange of τ and σ derivatives. The boundary conditions (2.10) are also invariant because $\gamma_\mu^0|_{\partial\tau \leftrightarrow \partial\sigma} = G_{\mu\nu}\ddot{q}^\nu - 2b_{\mu\nu}q'^\nu - 2h_{\mu\nu}(q)q'^\nu$ is identically equal to zero at $\sigma = 0$. Therefore, we can conclude that the two ansatz equations (2.11) must be invariant too. It follows that $A_{1\nu}^\mu = A_{2\nu}^\mu \equiv A_\nu^\mu$ and $\beta_{1\nu}^\mu = \beta_{2\nu}^\mu \equiv \beta_\nu^\mu$. Moreover, comparing (2.11) with (2.12), we conclude that A_ν^μ is infinitesimal.

Now, substituting (2.11) with the redefined coefficients into the boundary condition (2.10) for $\sigma = 0$, we obtain

$$\left[G\beta(q)\right]_{\mu\nu}\Big|_{\sigma=0} = B_{\mu\nu}(q)\Big|_{\sigma=0}, \quad (2.13)$$

which gives $\beta_\nu^\mu(q) = (G^{-1})^{\mu\rho}B_{\rho\nu}(q)$. The ansatz (2.11) transforms to

$$\begin{aligned} \ddot{q}^\mu &= -A_\nu^\mu(\tilde{q})\dot{q}^\nu + 2\left[G^{-1}B(q)\right]_\nu^\mu q'^\nu, \\ \dot{q}'^\mu &= -A_\nu^\mu(\tilde{q})q'^\nu + 2\left[G^{-1}B(q)\right]_\nu^\mu \dot{q}^\nu. \end{aligned} \quad (2.14)$$

Substituting these expressions into the equations of motion (2.9), we obtain

$$\ddot{q}^\mu - q''^\mu = 12[h_\nu^\mu(\dot{q})b_\rho^\nu\dot{q}^\rho - h_\nu^\mu(q')b_\rho^\nu q'^\rho], \quad (2.15)$$

and

$$\dot{A}_\nu^\mu(\tilde{q})\dot{q}^\nu - A_\nu^\mu(\tilde{q})q'^\nu + A_\nu^\mu(\tilde{q})(\ddot{q}^\nu - q''^\nu) = 2h_\nu^\mu\dot{q}^\nu - 24h_\nu^\mu(bq)(b\dot{q})^\nu, \quad (2.16)$$

and the consistency relation $(\dot{q}^\mu)' = (\dot{q}'^\mu)$ gives

$$2B_\nu^\mu(q)(\ddot{q}^\nu - q''^\nu) = \dot{A}_\nu^\mu(\tilde{q})q'^\nu - A_\nu^\mu(\tilde{q})\dot{q}^\nu. \quad (2.17)$$

We can eliminate $\ddot{q}^\nu - q''^\nu$ from (2.16) and (2.17), using (2.15) to obtain two equations for the unknown coefficient A_ν^μ

$$\dot{A}_\nu^\mu(\tilde{q})\dot{q}^\nu - A_\nu^\mu(\tilde{q})q'^\nu = 2h_\nu^\mu\dot{q}^\nu - 24h_\nu^\mu(bq)(b\dot{q})^\nu, \quad (2.18)$$

$$\dot{A}_\nu^\mu(\tilde{q})q'^\nu - A_\nu^\mu(\tilde{q})\dot{q}^\nu = 24\left[b\dot{h}b\dot{q} - bh'bq'\right]^\mu. \quad (2.19)$$

Note that the third term at the left hand side of (2.16) has been neglected, being the higher order term as the product of the infinitesimally small A_ν^μ and $\ddot{q}^\nu - q''^\nu$.

Because in all relations (equations of motions, boundary conditions and ansatz) the sum of σ and τ derivatives is preserved and \tilde{q} is odd, we conclude

$$\dot{\tilde{q}}^\mu = q'^\mu, \quad \tilde{q}'^\mu = \dot{q}^\mu. \quad (2.20)$$

The interpretation of these equations will be discussed at section 3. From (2.20) follows

$$\dot{A}_\nu^\mu(\tilde{q}) = A_\nu^\mu(\dot{\tilde{q}}) = A_\nu^\mu(q'), \quad A_\nu^\mu(\tilde{q}) = A_\nu^\mu(\tilde{q}') = A_\nu^\mu(\dot{q}), \quad (2.21)$$

and the solutions of (2.18) and (2.19) are

$$\begin{aligned} {}_{(1)}A^\mu{}_\nu(q) &= (G^{-1})^{\mu\rho} \left[h(q) - 12h(bq)b \right]_{\rho\nu}, \\ {}_{(2)}A^\mu{}_\nu(q) &= (G^{-1})^{\mu\rho} \left[-12bh(q)b + 12bh(bq) \right]_{\rho\nu}. \end{aligned} \quad (2.22)$$

Because the solution ${}_{(1)}A^\mu{}_\nu$ satisfies the homogeneous part of the equation (2.19) and the solution ${}_{(2)}A^\mu{}_\nu$ satisfies the homogeneous part of the equation (2.18), the complete solution for $A^\mu{}_\nu$ is of the form

$$A^\mu{}_\nu(q) = (G^{-1})^{\mu\rho} \left[h(q) - 12bh(q)b - 12h(bq)b + 12bh(bq) \right]_{\rho\nu}, \quad (2.23)$$

with the property $(GA)_{\mu\nu} = -(GA)_{\nu\mu}$. Finally, we can write the space-time coordinates satisfying the boundary condition at $\sigma = 0$ as

$$\begin{aligned} \dot{x}^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})] \dot{q}^\nu + 2[G^{-1}B(q)]^\mu{}_\nu q'^\nu, \\ x'^\mu &= [\delta^\mu_\nu - A^\mu{}_\nu(\tilde{q})] q'^\nu + 2[G^{-1}B(q)]^\mu{}_\nu \dot{q}^\nu. \end{aligned} \quad (2.24)$$

For $\sigma = \pi$ instead of (2.8), we define even and odd variables with respect to $\sigma = \pi$

$$\begin{aligned} {}^*q^\mu(\sigma) &= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu} \Big|_{\sigma=\pi}, \\ {}^*\bar{q}^\mu(\sigma) &= - \sum_{n=0}^{\infty} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu} \Big|_{\sigma=\pi}. \end{aligned} \quad (2.25)$$

Applying the analogous procedure as for the case $\sigma = 0$, we obtain

$$\begin{aligned} \dot{x}^\mu(\sigma) &= \left[\delta^\mu_\nu - A^\mu{}_\nu[{}^*\tilde{q}(\pi - \sigma)] \right] {}^*\dot{q}^\nu(\pi - \sigma) + 2 \left[G^{-1}B[{}^*q(\pi - \sigma)] \right]^\mu{}_\nu {}^*q'^\nu(\pi - \sigma), \\ x'^\mu(\sigma) &= \left[\delta^\mu_\nu - A^\mu{}_\nu[{}^*\tilde{q}(\pi - \sigma)] \right] {}^*q'^\nu(\pi - \sigma) + 2 \left[G^{-1}B[{}^*q(\pi - \sigma)] \right]^\mu{}_\nu {}^*\dot{q}^\nu(\pi - \sigma). \end{aligned} \quad (2.26)$$

Note that if

$$q^\mu(\sigma) = {}^*q^\mu(\pi - \sigma), \quad \bar{q}^\mu(\sigma) = {}^*\bar{q}^\mu(\pi - \sigma), \quad (2.27)$$

then the solutions (2.24) and (2.26) coincide, and from the relation (2.27) follows the 2π -periodicity of x^μ . So, if we extend the σ domain and demand 2π -periodicity of the original variable $x^\mu(\sigma + 2\pi) = x^\mu(\sigma)$, the relation (2.24) solves both constraints at $\sigma = 0$ and $\sigma = \pi$.

Let us stress that the solution of the boundary condition does not depend on one effective variable only, but on the two variables q^μ and \tilde{q}^μ , connected by the relation (2.20). So, we obtained some non-geometric space with the doubled number of degrees of freedom but with the constraint (2.20).

2.2 Effective theory

Substituting the solution (2.24) into the Lagrangian (2.1) we obtain the effective Lagrangian

$$\mathcal{L}^{eff} = \frac{\kappa}{2} \dot{q}^\mu G_{\mu\nu}^{eff}(q, \tilde{q}) \dot{q}^\nu - \frac{\kappa}{2} q'^\mu G_{\mu\nu}^{eff}(q, \tilde{q}) q'^\nu + 2\kappa q'^\mu B_{\mu\nu}^{eff}(q, \tilde{q}) \dot{q}^\nu, \quad (2.28)$$

where

$$\begin{aligned} G_{\mu\nu}^{eff}(q, \tilde{q}) &= G_{\mu\nu}^E(q + 2b\tilde{q}) + 4[b^2 A(\tilde{q}) - A(\tilde{q})b^2]_{\mu\nu}, \\ B_{\mu\nu}^{eff}(q, \tilde{q}) &= [h(2b\tilde{q}) + 4bh(2b\tilde{q})b]_{\mu\nu} - B_\mu{}^\rho(q) G_{\rho\nu}^E(q). \end{aligned} \quad (2.29)$$

The expression

$$G_{\mu\nu}^E(x) \equiv G_{\mu\nu} - 4B_{\mu\rho}(x)(G^{-1})^{\rho\sigma} B_{\sigma\nu}(x) \quad (2.30)$$

is the definition of the open string metric, which was introduced in [12] for the constant background case. In our case of the weakly curved background (in the leading order in $B_{\mu\nu\rho}$) it is equal to

$$\begin{aligned} G_{\mu\nu}^E &= g_{\mu\nu} - 4(bh + hb)_{\mu\nu}, \\ g_{\mu\nu} &= G_{\mu\nu} - 4(bG^{-1}b)_{\mu\nu} = G_{\mu\nu}^E(0). \end{aligned} \quad (2.31)$$

Because our basic variable $q^\mu(\sigma)$ contains only even powers of σ , it is convenient to regard it as the even function $q^\mu(-\sigma) = q^\mu(\sigma)$ on the interval $\sigma \in [-\pi, \pi]$. Hereafter, we will consider the action $S^{eff} = \int d\tau \int_{-\pi}^{\pi} d\sigma \mathcal{L}^{eff}$, and consequently, the terms of the effective metric which depend on \tilde{q} and the term of effective Kalb-Ramond field which depends on q will disappear, so that

$$S^{eff} = \kappa \int_{\Sigma_1} d^2\xi \left[\frac{1}{2} \eta^{\alpha\beta} G_{\mu\nu}^{eff}(q) + \epsilon^{\alpha\beta} B_{\mu\nu}^{eff}(2b\tilde{q}) \right] \partial_\alpha q^\mu \partial_\beta q^\nu. \quad (2.32)$$

Here Σ_1 marks the changed sigma domain $\sigma \in [-\pi, \pi]$. The effective background fields are equal to

$$\begin{aligned} G_{\mu\nu}^{eff}(q) &= G_{\mu\nu}^E(q), \\ B_{\mu\nu}^{eff}(2b\tilde{q}) &= -\frac{\kappa}{2} [g\Delta\theta(2b\tilde{q})g]_{\mu\nu}, \end{aligned} \quad (2.33)$$

where $\Delta\theta$ is the infinitesimal part of the so called non-commutativity parameter

$$\theta^{\mu\nu} = -\frac{2}{\kappa} [G_E^{-1} B G^{-1}]^{\mu\nu} = \theta_0^{\mu\nu} - \frac{2}{\kappa} [g^{-1}(h + 4bh b)g^{-1}]^{\mu\nu}. \quad (2.34)$$

It is defined in analogy with that of the flat space-time introduced in [12]. The constant parts of the effective metric and the non-commutativity parameter are denoted by $g_{\mu\nu} = G_{\mu\nu}^E(0)$ and $\theta_0^{\mu\nu} = \theta^{\mu\nu}(0) = -\frac{2}{\kappa} [g^{-1} b G^{-1}]^{\mu\nu}$.

3 Doubled geometry of the effective theory

There are two somewhat unexpected things in the effective theory (2.32). The first one is the appearance of the non-trivial Kalb-Ramond field $B_{\mu\nu}^{eff}$. The second one, which is essentially the cause of the first, is the fact that it does not depend on the coordinate q^μ but on \tilde{q}^μ . Let us analyze these results pursuing an explanation and interpretation.

It is well known that the theory of the unoriented closed string, which is for example the effective theory for the constant background, does not contain the Kalb-Ramond field. Standardly, this is explained [9, 15] by noticing that the effective Kalb-Ramond field appears in the effective action within the term $B_{\mu\nu}^{eff}\dot{q}^\mu q'^\nu$. If the Kalb-Ramond field depends on the Ω -even variable, this term does not contribute.

In the case of the weakly curved background the effective Kalb-Ramond field does not depend on the effective coordinate q^μ (Ω -even) but on \tilde{q}^μ (Ω -odd). Since, $B_{\mu\nu}^{eff}(2b\tilde{q})$ is proportional to \tilde{q}^μ it means that the effective Kalb-Ramond field is odd under σ -parity transformation $\Omega B_{\mu\nu}^{eff}[2b\tilde{q}(\sigma)] = -B_{\mu\nu}^{eff}[2b\tilde{q}(\sigma)]$. This makes the term $B_{\mu\nu}^{eff}(2b\tilde{q})\dot{q}^\mu q'^\nu$ Ω -even and allows its survival.

To find the interpretation of \tilde{q}^μ , let us solve the relations in (2.20). First, note that \tilde{q}^μ appears as an argument of $B_{\mu\nu}^{eff}$ only, which is the infinitesimal of the first order. So, it is enough to consider \tilde{q}^μ in the zeroth order. The zeroth order equation of motion for q^μ is just $\partial_+\partial_-q^\mu = 0$, and consequently the solution has the form $q^\mu(\sigma) = f^\mu(\sigma^+) + g^\mu(\sigma^-)$, with $\sigma^\pm = \tau \pm \sigma$. The Ω -evenness of the variable q^μ , $q^\mu(-\sigma) = q^\mu(\sigma)$, implies $f(\sigma) = g(\sigma)$ and we obtain

$$q^\mu(\sigma) = f^\mu(\sigma^+) + f^\mu(\sigma^-). \quad (3.1)$$

From the properties $\partial_\pm f^\mu(\sigma^\mp) = 0$, we have $\dot{f}^\mu(\sigma^\pm) = \pm f'^\mu(\sigma^\pm)$. Therefore, $\dot{q}^\mu(\sigma) = f'^\mu(\sigma^+) - f'^\mu(\sigma^-)$, and consequently for both equations from (2.20) we obtain

$$\tilde{q}^\mu(\tau, \sigma) = f^\mu(\sigma^+) - f^\mu(\sigma^-) + f_0. \quad (3.2)$$

The variable \tilde{q}^μ is odd by definition, so it can not contain the constant part, therefore the integration constant f_0 in (3.2) is zero. This means that $\tilde{q}^\mu(\tau, \sigma)$ is T-dual mapping of the effective coordinate $q^\mu(\tau, \sigma)$ (see for example (17.76) of Ref. [9] or eq. (6.17) of Ref.[10]).

Let us comment on the above conclusion. The Buscher construction of the T-dual theory, done for the constant background, can be generalized to the case of the weakly curved background Ref. [17]. The dual coordinate y_μ can be expressed in terms of the original one x^μ as

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}[x]\partial_\pm x^\nu \mp 2\beta_\mu^\mp[x], \quad (3.3)$$

where $\Pi_{\pm\mu\nu} \equiv B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ and $\beta_\mu^\alpha[x] \equiv \partial_\mu B_{\nu\rho}\epsilon^{\alpha\beta}x^\nu\partial_\beta x^\rho$ is infinitesimal. In the present article, we discuss the T-dual coordinate of q^μ , the even part of coordinate x^μ . So, the role of the initial coordinate x^μ takes q^μ , and the role of the dual coordinate y_μ takes \tilde{q}^μ .

Projecting (3.3) into odd and even part and neglecting first order in $B_{\mu\nu\rho}$ terms, because the dual coordinate is the argument of the infinitesimally small A^μ_ν , one obtains

$$\dot{\tilde{q}}^\mu \cong q'^\mu, \quad \tilde{q}'^\mu \cong \dot{q}^\mu. \quad (3.4)$$

These are just the expressions (2.20). So, q^μ and \tilde{q}^μ are not just duals in the zeroth order but truly the dual coordinates.

So, in the effective theory, the effective metric depends on the effective coordinate q^μ and the effective Kalb-Ramond field on its T-dual \tilde{q}^μ . This kind of background is seen to be possible in the so called doubled formulation (the analysis of the nongeometrical backgrounds where the T-duality is allowed as the transition function). The relations (2.20) we just solved, represent the self duality constraint in terminology of Ref. [1] which reduces the number of the degrees of freedom by half in the doubled non-geometrical space of q^μ and \tilde{q}^μ .

4 The effective Kalb-Ramond field as a torsion potential

The effective theory (2.32) is defined on the nongeometrical space (q^μ, \tilde{q}^μ) where the second coordinate \tilde{q}^μ is the T-dual of the first effective coordinate q^μ . It is interesting to find the equation of motion of the theory of this type, when $G_{\mu\nu}^{eff}$ is arbitrary function of q^μ and $B_{\mu\nu}^{eff}$ consists of the odd powers in Ω -odd variable $\bar{q} = 2b\tilde{q}$.

Even-though the effective theory is defined in the doubled space with coordinates q^μ and \tilde{q}^μ , these are related by the self-duality constraint (2.20). So, to obtain the equations of motion, we will vary the action with respect to the independent variable q^μ . They contain the non-local part which is obtained by varying with respect to \tilde{q} . The form of non-local term is a consequence of the relation $\delta\dot{q} = \delta\tilde{q}'$, which follows from (2.20).

The variation produces

$$\eta^{\alpha\beta} {}^0D_\beta \partial_\alpha q^\mu = 2\kappa\theta_0^{\mu\nu} \epsilon^{\alpha\beta} \int_0^\sigma d\eta \frac{\partial}{\partial\tau} \left[\frac{\partial B_{\rho\sigma}^{eff}(\bar{q})}{\partial\bar{q}^\nu} \partial_\alpha q^\rho \partial_\beta q^\sigma \right], \quad (4.1)$$

where ${}^0D_\alpha V^\mu = \partial_\alpha q^\nu {}^0D_\nu V^\mu$ is the covariant derivative in the world-sheet direction. ${}^0D_\mu$ is the generalized covariant derivative ${}^0D_\mu V^\nu = \partial_\mu V^\nu + {}^0\Gamma_{\rho\mu}^\nu V^\rho$, with the generalized connection

$${}^0\Gamma_{\rho\sigma}^\mu = (\Gamma_{eff})_{\rho\sigma}^\mu + \frac{1}{2} {}^0K_{\rho\sigma}^\mu. \quad (4.2)$$

It consists of the Christoffel connection $(\Gamma_{eff})_{\rho\sigma}^\mu$ and contorsion ${}^0K_{\mu\rho\sigma} = \frac{1}{2} {}^0T_{\{\sigma\mu\rho\}}$ in terms of torsion

$${}^0T_{\mu\rho\sigma} = 4b_\rho{}^\nu \frac{\partial}{\partial\bar{q}^\nu} B_{\mu\sigma}^{eff}. \quad (4.3)$$

Here, $\{\mu\nu\rho\} = \nu\rho\mu + \rho\mu\nu - \mu\nu\rho$ is Schouten bracket.

Therefore, the Kalb-Ramond field is related to the torsion potential. This is in accordance with the usual interpretation of the Kalb-Ramond field $B_{\mu\nu}$ in the low energy string theory as the non-Rimannian theory [16]. In our case torsion is an infinitesimally small constant.

5 Conclusion

In the present paper we demonstrated that the solution of the Neumann boundary conditions on the open string endpoints in the weakly curved background, naturally leads to the effective theory in the non-geometric background. This background depends both on the conventional effective coordinate q^μ and on its T-dual \tilde{q}^μ .

We started with the oriented open string theory, and solved the Neumann boundary conditions. We presupposed the form of the solution and found its explicit expression. On the solution of the boundary conditions, the theory reduced to the effective theory. In the case of the constant background fields, the effective theory turns to be a closed string theory defined on the unoriented orbifold, because the effective coordinate is symmetric under σ -parity transformation, $\Omega : \sigma \rightarrow -\sigma$ and satisfies the boundary condition $q^\mu(\sigma = -\pi) = q^\mu(\sigma = \pi)$.

In our case, the complete transition from the original theory (2.1) to the effective theory (2.32) consists of

1. the transition from conventional to the *doubled geometry*

$$x^\mu \rightarrow q^\mu, \tilde{q}^\mu \quad (5.1)$$

2. *the background field transition*

$$G_{\mu\nu} \rightarrow G_{\mu\nu}^{eff}(q), \quad B_{\mu\nu}(x) \rightarrow B_{\mu\nu}^{eff}(2b\tilde{q}). \quad (5.2)$$

The first transition says that while the original theory is defined on the geometric target space, the effective one is defined on the enlarged, the so called doubled target space, given in terms of both the effective coordinate q^μ and its T-dual \tilde{q}^μ . Such a space is non-geometrical space [1]. In our case of the weakly curved background, the non-geometrical space arose naturally in the solution of the Neumann boundary conditions.

The appearance of the doubled target space allowed the string to see the effective background field $B_{\mu\nu}^{eff}$. In fact, the effective theory is Ω -even projection of the initial one, therefore in the geometric background the term with Kalb-Ramond field vanished as the Ω -odd term in the action. But, in doubled target space Kalb-Ramond field depends on the T-dual coordinate \tilde{q}^μ , which is Ω -odd. So, the corresponding term in the action becomes Ω -even and can not be projected out by the world-sheet parity projection.

Let us summarize the forms of the initial and the corresponding effective backgrounds in the following table.

| <i>Original metric $G_{\mu\nu}$</i> | <i>Original KR field $B_{\mu\nu}$</i> |
|---|---|
| <i>Effective metric $G_{\mu\nu}^{eff}$</i> | <i>Effective KR field $B_{\mu\nu}^{eff}$</i> |
| $G_{\mu\nu}$ | $b_{\mu\nu}$ |
| $g_{\mu\nu} = G_{\mu\nu} - 4b_{\mu\rho}b^\rho_\nu$ | 0 |
| $G_{\mu\nu}$ | $\frac{1}{3}B_{\mu\nu\rho}x^\rho$ |
| $G_{\mu\nu}$ | 0 |
| $G_{\mu\nu}$ | $b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$ |
| $G_{\mu\nu} - 4B_{\mu\rho}(q)B^\rho_\nu(q)$ | $-\frac{\kappa}{2}(g\Delta\theta(2b\tilde{q})g)_{\mu\nu}$ |

The doubled target space and nontrivial Kalb-Ramond field appear only in the third row for the weakly curved background of the original theory.

In contrast to the great majority of papers where the flat constant background is assumed, in the present paper we investigated the string in the coordinate dependent background. Unexpectedly, the effective theory obtained in this case, significantly differs from the one obtained in the flat background. Here, not only that the effective background is not constant, it can not be described by the single effective coordinate q^μ , but one needs the additional variable \tilde{q}^μ . So, considering the problem of solving the boundary conditions in the curved background, the doubled space appears naturally. We also derived the equation of motion in the doubled space, which is nontrivial because variables q^μ and \tilde{q}^μ are not independent.

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